Thm A. *The quotient ring* $\mathbf{R}[x]/\langle(x^2+1)\rangle$ *is isomorphic to the field* $\mathbf{C}$.

Proof: Consider the map $\psi : \mathbf{R}[x] \to \mathbf{C}$ defined by $\psi\big(p(x)\big) = p(i)$.

Since $\psi(a+bx) = a+bi$ (for any real numbers a and b) it is clear that ψ is surjective.

Since $\psi(p(x)+q(x)) = p(i)+q(i) = \psi(p(x))+\psi(q(x))$ and $\psi(p(x)q(x)) = p(i)q(i) = \psi(p(x))\psi(q(x))$, it follows that ψ is a homomorphism.

Now the kernel of ψ is the set of all polynomials $p(x) \in \mathbf{R}[x]$ such that $p(i) = 0$. Note that (for real polynomials) $p(i) = 0$ iff $p(-i) = 0$ (since *complex* roots of *real* polynomials always come in conjugate pairs). Thus $p(i) = 0$ iff $p(x)$ is divisible by $(x-i)$ and $(x+i)$ (by the Factor Fheorem). Since $(x-i)$ and $(x+i)$ are irreducible, any polynomial divisible by both of them is divisible by their product $(x^2-i^2) = (x^2+1)$. Thus the kernel of ψ is just the set $\langle(x^2+1)\rangle$ of polynomials divisible by x^2+1.

By the First Isomorphism Theorem, we have proved that $\mathbf{R}[x]/\langle(x^2+1)\rangle$ is isomorphic to the field of complex numbers. □

Thm B. *The quotient ring* $\mathbf{R}[x]/\langle(x^2-1)\rangle$ *is isomorphic to the ring* $\mathbf{R}\times\mathbf{R}$.

Proof: Consider the map $\chi : \mathbf{R}[x] \to \mathbf{R}\times\mathbf{R}$ defined by $\chi\big(p(x)\big) = \big(p(1), p(-1)\big)$.

Since $\chi\Big(\frac{a+b}{2} + \frac{a-b}{2}x\Big) = (a,b)$ (for any real numbers a and b) it is clear that χ is surjective.

Since $\chi(p(x)+q(x)) = \big(p(1)+q(1), p(-1)+q(-1)\big) = \big(p(1), p(-1)\big)+\big(p(-1), q(-1)\big) = \chi(p(x)) + chi(q(x))$ it is clear that χ is a homomorphism.

Now the kernel of ψ is the set of all polynomials $p(x) \in \mathbf{R}[x]$ such that $p(1) = 0$ and $p(-1) = 0$ (since the zero element in the ring $\mathbf{R}\times\mathbf{R}$ is $(0,0)$). Thus (again by the Factor Theorem), the kernel is the set of polynomials which are divisible by both $x-1$ and $x+1$. In other words, the kernel is the set of polynomials in $\mathbf{R}[x]$ which are multiples of x^2-1, namely the ideal $\langle(x^2-1)\rangle$.

By the First Isomorphism Theorem, we have proved that $\mathbf{R}[x]/\langle(x^2-1)\rangle$ is isomorphic to the ring $\mathbf{R}\times\mathbf{R}$. □

Thm C. *Let* I *be the ideal in* $\mathbf{Z}[x]$ *defined by* $I = \{p(x) \in \mathbf{Z}[x] \,\big|\, p(0) \in 2\mathbf{Z}\}$. *Then* $\mathbf{Z}[x]/I$ *is isomorphic to the field* $\mathbf{Z}_2$. *Thus* I *is a maximal ideal in* $\mathbf{Z}[x]$.

Proof: Note that $\mathbf{Z}_2$ is shorthand for $\mathbf{Z}/2\mathbf{Z}$.

Let $\alpha : \mathbf{Z}[x] \to \mathbf{Z}$ be defined by $\alpha\big(p(x)\big) = p(0)$, and let $\beta : \mathbf{Z} \to \mathbf{Z}_2$ be the natural homomorphism defined by $\beta(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$. Since α is easily shown to be a homomorphism (and we already know that the natural map β is a homomorphism), the composition $\beta\circ\alpha$ is a homomorphism from $\mathbf{Z}[x]$ to $\mathbf{Z}_2$. Clearly $\beta\circ\alpha$ is surjective.

Now the kernel of α is $2\mathbf{Z}$ so the kernel of $\beta\circ\alpha$ is the set of polynomials $\{p(x)\,\big|\,\alpha(p(x)) \in 2\mathbf{Z}\} = \{p(x)\,\big|\,p(0) \in 2\mathbf{Z}\} = I$. Thus we have met the conditions for the First Isomorphism Theorem (using the composition homomorphism $\beta\circ\alpha$), and proved the desired result. □